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RESEARCH REPORT No. EM-55

ASYMPTOTIC EVALUATION OF THE FIELD AT A CAUSTIC

by

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and

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P. 16 — l. 7

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New York University
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I. Introduction

The electromagnetic field which can be constructed by the methods of geometrical optics, following R.K. Luneburg [1], becomes infinite at caustics of the optical ray systems. In this paper we overcome this difficulty and obtain a finite value for the field at a caustic which is produced by the reflection of any incident wave from an arbitrary reflector (all in two dimensions for simplicity). What we actually determine is the leading term in the asymptotic expansion of the field with respect to $k = \frac{2\pi}{\lambda}$, for large k. Off the caustic this is just the geometrical optics field, but on the caustic it is proportional to $k^{\frac{1}{2} - \frac{1}{n}}$ where n is an integer greater than two. At a perfect focus, such as the focus of a parabolic cylinder, the field is proportional to $k^{\frac{1}{2}}$. Thus on a caustic or at a focus the field increases indefinitely as k does.

We have exemplified these general results by considering the special cases of a plane wave incident on the concave sides of a segment of a parabolic cylinder and a segment of a circular cylinder. In the parabolic case, with incidence along the axis of the parabola, the entire caustic is a point, the focus of the parabola. The asymptotic behavior of the field at the focus is obtained explicitly and transition formulae which show how the field varies in the neighborhood of the focus are also obtained. These formulae involve integrals which we have evaluated numerically in some special cases.

In the case of the circular cylinder, the caustic is a curve with a cusp. The asymptotic behavior at the cusp is different from that on the rest of the caustic. The variation of the field near the caustic is described in terms of Hardy's generalized Airy functions.

Since our results yield exactly the leading term in the asymptotic expansion of the field, they constitute a check on the solutions proposed by Debye [2] Picht [3] and Luneburg [4]. Each

of these authors constructed a solution of the wave equation, or of Maxwell's equations, which is regular in an infinite homogeneous space and satisfies conditions at infinity. Luneburg proved that the solution thus obtained reduces to the known geometrical optics solution as k becomes infinite. In an actual problem involving boundaries the correct solution must satisfy certain boundary conditions. Since the solutions just mentioned do not satisfy such conditions, it is not certain that they will yield the correct field at a caustic in a real problem involving boundaries. However, we have proved that they do yield exactly the same asymptotic field on a caustic as is given by our method, and therefore their use in practical problems is justified.

II. Formulation: The Geometrical Optics Field

We consider the solution $E(x,y)$ of the reduced wave equation

(1) $$(\nabla^2 + k^2) E = 0.$$

The solution must satisfy the boundary conditions

(2) $$E = 0 \quad \text{on } C .$$

Here C is a given piecewise smooth curve which we will call a reflector.* At any corners of C the derivatives of E may become singular. Therefore we must impose a regularity condition at these corners, e.g. that the singularity be the weakest possible one, or that the "energy" integral be finite. In addition we assume that an incident field $E_i(x,y)$ which satisfies (1) is given, and that the reflected field $E_r = E - E_i$ satisfies the radiation condition

(3) $$E_r = E - E_i \quad \text{is outgoing.}$$

These conditions complete the formulation of the problem. This formulation applies to the z component of the electric field when the reflector is a perfect conductor and the surrounding medium is homogeneous.

Rather than find E itself, we treat the simpler problem of determining the leading term in the asymptotic expansion of E with respect to k, for large k. This is what we call the geometrical optics field. We begin with the leading term in the asymptotic expansion of the incident field E_i. We assume it to be of the form

(4) $$E_i \sim A_i(x,y) \exp\left[ik\psi_i(x,y)\right] .$$

Now, following Luneburg, we know that the zero-order term in the asymptotic expansion of the field is zero in the shadow. Furthermore, the leading term in the reflected field is

(5) $$E_r \sim A_r(x,y) \exp\left[ik\psi_r(x,y)\right] .$$

The last statement must be qualified in two respects. First, the form (5) applies only at points not on caustics or shadow boundaries. Second, the leading term will be a sum of expressions of the type given in (5) at points through which more than one reflected ray passes. There will be one such summand for each ray.

---

* If C is closed, we consider the solution outside C only, and if C extends to infinity dividing the plane into two parts, we consider the solution in one part only.

If we apply (2) to the asymptotic form of E we obtain

(6) $E \sim A_i(x,y) \exp\left[ik\psi_i(x,y)\right] + A_r(x,y) \exp\left[ik\psi_r(x,y)\right] = 0$ on C.

From (6) we have

(7) $$A_r(x,y) = - A_i(x,y) \quad \text{on } C$$

(8) $$\psi_r(x,y) = \psi_i(x,y) \quad \text{on } C.$$

(If E_r consists of a sum of terms, the remaining terms occur in pairs satisfying (7),(8), where the indices i and r refer to respective terms of the pair.)

A_r and ψ_r off C can be determined by the methods of geometrical optics, using the values on C given by (7), (8). In particular, ψ_r satisfies the eiconal equation

(9) $$|\nabla \psi_r| = 1.$$

Once ψ_r is found, A_r can be constructed by the conservation of energy formula

(10) $$A_r(x,y) = \left(\frac{d\sigma_o}{d\sigma}\right)^{1/2} A_r(x_o,y_o) \, .$$

Here x,y and x_o,y_o are two points on a reflected ray, and $d\sigma_o$, $d\sigma$ respectively are cross-sectional lengths of an infinitesimal tube of rays at each of these points. If we choose for x_o,y_o a point on C, then (10) determines A_r off C. If D is the distance along the ray from x_o,y_o to x,y and if K_r is the curvature of the reflected wavefront at x_o,y_o, then it is easily shown that

(11) $$\frac{d\sigma_o}{d\sigma}^{1/2} = (1 - DK_r)^{-1/2} \, .$$

On the basis of (5)-(11) the asymptotic form of the reflected field may be immediately written as

(12) $$E_r(x,y) \sim -(1 - DK_r)^{-1/2} A_i(x_o,y_o) \exp\left[ik\psi_i(x_o,y_o) + ikD\right] \, .$$

The point x_o,y_o is the point on C from which the ray through x,y is reflected, and D is the distance between these two points. If there is no reflected ray through (x,y) then $E_r(x,y) \sim 0$.

When $DK_o = 1$, A_r becomes infinite. This occurs at the caustics of the reflected rays.

III The Field at a Caustic

In order to obtain a finite value for E_r at a caustic, we begin with Green's formula

$$(13) \qquad E(x,y) = E_i(x,y) + \frac{1}{4i} \int_C \frac{\partial E}{\partial n} H_o^{(1)}(kD)\, ds .$$

Here D is the distance from the integration point x_o, y_o on C to x, y, and $H_o^{(1)}$ is the zero-order Hankel function of the first kind. To obtain the asymptotic form of E at any point, and in particular at or near a caustic, we insert into the integrand the asymptotic form of $\frac{\partial E}{\partial n}$, which can be computed from (4) and (12). We also replace $H_o^{(1)}$ by its asymptotic form and then evaluate the integral asymptotically. In this way we get back the expression for E_r given by (12) at all points off the caustic, as we must expect. At the caustic, however, we obtain a finite result rather than the infinite value given by (12).

To justify the method described we must assume that the caustic does not intersect C. Furthermore the lower-order terms in the expansion of $\frac{\partial E}{\partial n}$, which have been neglected, must not contribute terms to the integral which are of the same (or higher) order as the contribution of the leading terms. That they do not can easily be proved to be the case if reasonable hypotheses are made about the omitted terms, but this will not be considered here.

We will now carry out the calculations just described. When $\frac{\partial E}{\partial n}$ is computed from (4) and (12) and the asymptotic form of $H_o^{(1)}$ is employed, equation (13) yields

$$(14) \quad E_r(x,y) \sim k^{1/2}(8\pi)^{-1/2} \int_C A_i\left[\frac{\partial \psi_i}{\partial n} - \frac{\partial \psi_r}{\partial n}\right] D^{-1/2} \exp\left[ik(\psi_i + D) - \frac{i\pi}{4}\right] ds .$$

This integral can be evaluated asymptotically by the method of stationary phase. The phase $\psi_i + D$ is a function of the integration point on C, and can be considered as a function of arc length s along C. The points of stationary phase are determined by

$$(15) \qquad \frac{d\psi_i}{ds} + \frac{dD}{ds} = 0 .$$

Then if $\frac{d^2}{ds^2}(\psi_i + D) \neq 0$ at the stationary point, (14) yields

$$(16) \qquad E_r(x,y) \sim \frac{A_i}{2}\left(\frac{\partial\psi_i}{\partial n} - \frac{\partial\psi_r}{\partial n}\right) D^{-1/2} \left|\frac{d^2}{ds^2}(\psi_i + D)\right|^{-1/2}$$

$$\cdot \exp\left[ik(\psi_i + D) - \frac{i\pi}{4}\left\{1 - \operatorname{sgn}\frac{d^2}{ds^2}(\psi_i + D)\right\}\right].$$

In (16) all quantities on the right side are to be evaluated at the point of stationary phase determined by (15). If there is no stationary point the right side is to be replaced by zero; if there is more than one stationary point, the right side is to be a sum with one term of the above type for each such point.

It follows from (15) that at the stationary points either the law of reflection is satisfied or the reflected ray is the continuation of the incident ray. In the latter case, evaluation of (16) shows that $E_r \sim -E_i$ so that $E \sim 0$. This occurs in the shadow. In the former case (16) is found to coincide with (12), as is to be expected. It is to be noted that the phase jumps by $\frac{\pi}{2}$ across the caustic.

The expression (16) is not valid when $\frac{d^2}{ds^2}(\psi_i + D) = 0$.

But if for a point x,y this second derivative is zero and also the first derivative (15) is zero, then it can be shown as follows that the point lies on a caustic of the reflected ray. $\psi_i + D$ is a function of x,y,s, say g(x,y,s).

For each value of s the equation g_s $(x,y,s) = 0$, which is (15), defines a ray. If g_{ss} $(x,y,s) = 0$ also, then the point x,y lies on the envelope of the family of rays, which is the caustic.

In order to determine the location of the caustic, we compute $\frac{d^2}{ds^2}(\psi_i + D)$ and obtain

$$(17) \qquad \frac{d^2}{ds^2}(\psi_i + D) = (D^{-1} + K_i)\cos^2\alpha - 2K_c \cos\alpha .$$

Here K_i and K_c are the curvatures of the incident wavefront and of the curve C, respectively, at the stationary point. The angle α is that between $\nabla\psi_i$ and the normal to C at the stationary point. The expression in (17) vanishes when D is given by (see also [5])

$$(18) \qquad D^* = (2K_c - K_i \cos\alpha)^{-1} \cos\alpha .$$

Thus (18) determines the distance D^* from C along a ray to the caustic, and thus the curvature K_r of the reflected wavefront at C is given by $K_r = (D^*)^{-1}$. The results (17),(18) are used in showing that (16) coincides with (12).

To obtain the field on the caustic we must evaluate the integral in (14) differently. For this purpose we consider the integral W given by

$$(19) \qquad W = \int A(s) \exp\left[ikf(s)\right] ds .$$

Suppose that at a stationary point $\bar{s}$ at which $f'(\bar{s}) = 0$ we also have

$$(20) \qquad f'(\bar{s}) = f''(\bar{s}) = \ldots = f^{(n-1)}(\bar{s}) = 0.$$

Then the Taylor series of $f(s)$ becomes

$$(21) \qquad f(s) = f(\bar{s}) + (s - \bar{s})^n \frac{f^{(n)}(\bar{s})}{n!} + \ldots..$$

Therefore W is asymptotically given by

$$(22) \qquad W \sim A(\bar{s}) \exp\left[ikf(\bar{s})\right] \int \exp\left[ik \frac{f^{(n)}(\bar{s})}{n!} (s - \bar{s})^n\right] ds,$$

Thus we finally obtain

$$(23) \qquad W \sim A(\bar{s}) \exp\left[ikf(\bar{s})\right] 2k^{\frac{-1}{n}} \Gamma(1 + n^{-1}) (n!)^{\frac{1}{n}} \left| f^{(n)}(\bar{s}) \right|^{\frac{-1}{n}} F_n ,$$

where

$$(24) \qquad \begin{aligned} F_n &= \exp\left[(i \tfrac{\pi}{2} n) \operatorname{sgn} f^{(n)}(\bar{s})\right] && \text{if } n \text{ is even} \\ &= \cos \tfrac{\pi}{2} n && \text{if } n \text{ is odd}. \end{aligned}$$

Let us apply the result (23) to the integral (14) when the point x,y lies on a caustic and when the first n-1 derivatives of $\psi_i + D$ vanish at the stationary point. Then we have

$$(25) \qquad E_r(x,y) \sim k^{\frac{1}{2} - \frac{1}{n}} (n!)^{\frac{1}{n}} \Gamma(1 + n^{-1}) (\pi D)^{\frac{-1}{2}} \sqrt{2} \left| \frac{d^n}{ds^n} (\psi_i + D) \right|^{\frac{-1}{n}} \cos\alpha$$

$$\cdot \exp\left[ik(\psi_i + D) - i \frac{\pi}{4}\right] F_n ,$$

where

$$
(26) \qquad
\begin{aligned}
F_n &= \exp\left[\left(i\frac{\pi}{2}n\right) \operatorname{sgn} \frac{d^n}{ds^n} (\psi_i + D)\right] && \text{if } n \text{ is even} \\
F_n &= \cos \pi/2n && \text{if } n \text{ is odd.}
\end{aligned}
$$

It is of interest to notice that the power of k which appears, namely $k^{\frac{1}{2} - \frac{1}{n}}$, is always a positive power on a caustic since $n \geq 3$. Thus the field increases indefinitely as k does. The larger the value of n the higher is the power of k, as one should expect because as n increases the phase of the integrand becomes more constant. If the phase is exactly constant, then the power $k^{1/2}$ occurs, as will be seen in the example of the parabolic cylinder.

IV. Comparison With the Debye - Picht - Luneburg Solution

In their treatments of the field at a focus or caustic, Debye, Picht and Luneburg each construct an exact solution of the wave equation, or of Maxwell's equations, which is regular in the infinite homogeneous space. Debye, Picht and Luneburg determine their solution by requiring it to have a specified behavior at infinity, and Luneburg proves that it must reduce to a specified geometrical optics field as k becomes infinite.

In most physical problems the field must satisfy boundary conditions on certain surfaces, and satisfy the wave equation or Maxwell's equations in a partly inhomogeneous medium. These conditions are not satisfied by the Debye - Picht - Luneburg solution. However, this solution does have the same behavior at infinity and the same geometrical optics field as the corresponding physical problem.

The question therefore arises as to whether the Debye - Picht - Luneburg solution implies that the field has the same asymptotic behavior at a caustic as that of the corresponding solution of the physical problem.

By the considerations of Section III (first and second paragraphs) both of these solutions, and in fact any solution with the correct geometrical optics field, will imply the correct behavior at caustics and foci pro-

vided that the lower-order terms in the integrand of (13) do not contribute higher-order terms. Thus all Debye - Picht - Luneburg solutions, although represented by less convenient integrals, will lead to the same leading asymtpotic term at a caustic as does equation (14).

V. Applications: 1. The Parabolic Cylinder

Consider a cylinder, the cross section of which is a segment of a parabola (see Figure 1). Let its focus be the origin of a Cartesian coordinate system, its axis be the x- axis, and its equation be

$$y^2 = -4p(x - p) \qquad x \geq \alpha \; . \tag{27}$$

As is indicated in the above equation, the segment of the parabola under consideration lies to the right of $x = \alpha$ and is concave toward the left, since we assume that the focal length p is positive.

We now suppose that a plane wave is incident from the left with its direction of propagation parallel to the x- axis. We assume that this wave is given by

$$E_i = e^{ikx} \; . \tag{28}$$

The incident rays, which are the orthogonal trajectories of the incident wavefronts x = constant, are obviously the lines y = constant parallel to the x- axis. Of these rays all those which intersect the parabolic segment will be reflected through the focus, and therefore the reflected wavefronts are segments of circles with the focus as origin. On the basis of this fact and equation (12), we have for the singly reflected field $E_r^{(1)}$

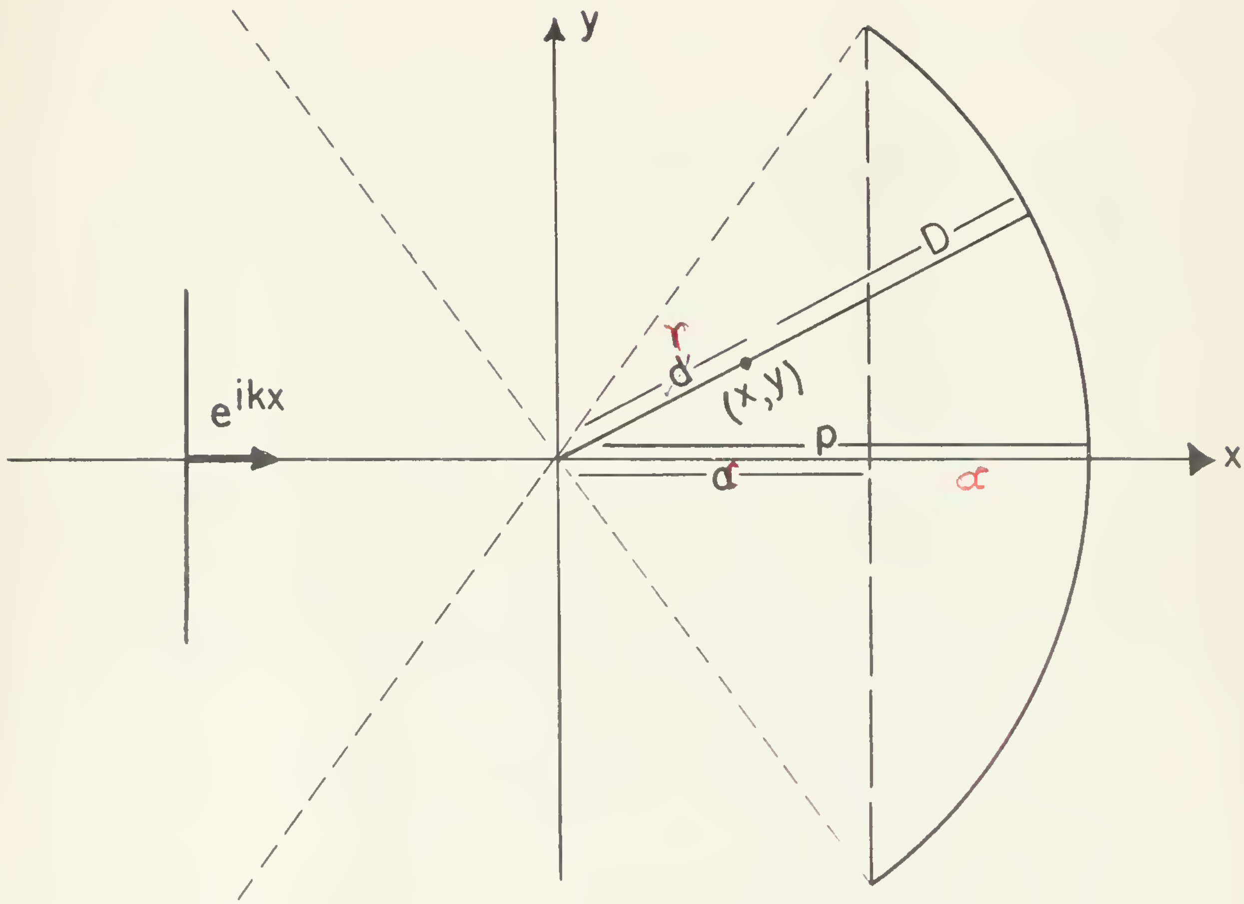

Figure 1

$$(29) \qquad E_r^{(1)}(x,y) \sim -\sqrt{\frac{D+r}{r}} \quad e^{ik(2p-r)} \qquad \text{before focus},$$

$$E_r^{(1)}(x,y) \sim -\sqrt{\frac{D-r}{r}} \quad e^{ik(2p+r)-i\pi/2} \qquad \text{after focus},$$

In equation (29) r is the distance of the point x,y from the focus and D is its distance along the reflected ray from the point of reflection on the parabola. The phase is easily determined since all rays reach the focus with the same phase $ik2p$.

Some rays intersect the parabolic segment a second time after having been reflected through the focus. In order to distinguish these rays, we introduce the angle θ between the once-reflected ray and the x- axis. We also introduce the angle γ γ between the x- axis and the line from the focus to the upper end of the parabolic segment. If $\gamma < \frac{\pi}{2}$ then $\alpha > 0$ and the entire segment lies to the right of the focus. In this case no ray is twice reflected. On the other hand, if we consider the complete parabola, then $\gamma = \pi$ and every ray is twice reflected, except the ray incident along the x- axis and characterized by $\theta = 0$. In all other cases ($\frac{\pi}{2} < \gamma < \pi$) all rays are twice reflected except those satisfying

$$(30) \qquad \pi - \gamma > \theta > -(\pi - \gamma),$$

which are once reflected.

After the second reflection, the reflected ray becomes parallel to the x- axis, and the doubly reflected field $E_r^{(2)}(x,y)$ is found from the equations (12) and (29) to be given by

$$(31) \qquad E_r^{(2)}(x,y) \sim \frac{2p}{y} \quad e^{ik(4p-x)-i\pi/2}.$$

The complete geometrical optics field can now be described as zero in the shadow, E_i at points outside the shadow through which no reflected ray passes, $E_i + E_r^{(1)}$ at points through which one reflected ray passes, $E_i + E_r^{(1)} + E_{r'}^{(1)}$ at points through which two singly reflected rays pass, $E_i + E_r^{(1)} + E_r^{(2)}$ at points through which one singly and one doubly reflected ray pass, and $E_i + E_r^{(1)} + E_{r'}^{(1)} + E_r^{(2)}$ at points through which two singly and one doubly reflected ray pass.

We now observe that the field $E_r^{(1)}$, as given by (29), becomes infinite at the focus of the parabola, and that the field $E_r^{(2)}$ as given by (31) becomes infinite along the axis $y = 0$. This latter singularity appears only in the case of the complete parabola, since the doubly reflected ray y = constant suffers it first reflecion at a point on the parabola with the ordinate $\frac{4p^2}{y}$. Thus as y approaches zero, the ordinate of the point of first reflection must become infinite, and this is possible only with the complete parabola. The x- axis is not a caustic in the ordinary sense of an envelope of rays, but the geometrical optics field becomes infinite there because the energy between incident rays with an arbitrarily large separation is reflected between rays with an arbitrarily small separation.

We will now assume that the parabolic segment is finite ($\gamma < \pi$), and we will obtain a finite expression for the field at and near the focus, which is the only caustic occurring in this case. To this end we employ (14), but we need not consider the contributions of $E_r^{(2)}$ to the integrand since the doubly reflected field contains no ray through the focus and therefore contributes only lower-order terms to the integral.

First we introduce the parabolic coordinates ξ, η defined by

$$\xi^2 = (x^2 + y^2)^{\frac{1}{2}} - x, \qquad \eta^2 = (x^2 + y^2)^{\frac{1}{2}} + x$$

(32)

$$x = \frac{1}{2}(\eta^2 - \xi^2), \qquad y = \xi\eta .$$

In these coordinates the equation of the parabolic segment is

$$(33)\qquad \eta = \pm\sqrt{2p}\,, \qquad 0 \le \xi \le \sqrt{2(p-\alpha)}\,.$$

Now (14) becomes

$$(34)\quad E_r^{(1)}(\xi,\eta) \sim -\sqrt{\frac{kp}{\pi}}\,\exp(ikp - i\tfrac{\pi}{4})\left\{\int_0^{\sqrt{2(p-\alpha)}} \left[D^{-\frac{1}{2}} \exp\left(ikD - \tfrac{1}{2}\xi_0^2\right)\right]_{\eta_0=\sqrt{2p}} d\xi_0\right.$$

$$\left. + \int_0^{\sqrt{2(p-\alpha)}} \left[D^{-\frac{1}{2}} \exp\left(ikD - \tfrac{1}{2}\xi_0^2\right)\right]_{\eta_0=-\sqrt{2p}} d\xi_0\right\}.$$

At the focus this yields

$$(35)\quad E_r^{(1)}(0,0) \sim -2\sqrt{\frac{kp}{\pi}}\,\exp(2ikp - i\tfrac{\pi}{4})\int_0^{\sqrt{2(p-\alpha)}} \left(\tfrac{1}{2}\xi_0^2 + p\right)^{-\frac{1}{2}} d\xi_0$$

$$\sim -2\sqrt{\frac{2kp}{\pi}}\,\exp(2ikp - i\tfrac{\pi}{4})\,\log\left[\sqrt{1-\frac{\alpha}{p}} + \sqrt{2-\frac{\alpha}{p}}\,\right].$$

To obtain the field near the focus we introduce the polar coordinates ρ, θ with the focus as origin and with ρ defined by $\rho = k(x^2 + y^2)^{\frac{1}{2}}$. Thus ρ is 2π times the distance measured in wavelengths, of a point from the focus. Now (34) becomes

$$(36) \qquad E_r^{(1)}(\rho,\theta) \sim -\sqrt{\frac{2kp}{\pi}}\ \exp\left(2\,ikp - \frac{i\pi}{4}\right)\int_{-\gamma}^{\gamma} \sec\frac{\phi}{2}\cdot e^{i\rho\cos(\theta+\phi)}\,d\phi \ .$$

Here $\gamma = \cos^{-1}\left[\frac{\alpha}{2p-\alpha}\right]$ is the angular coordinate of the end of the parabolic segment. Figures 2-4, based on eq. (36), show the variation of the reflected field amplitude $|E_r^{(1)}|$ along and at right angles to the axis for two different parabolic segments.

2. The Circular Cylinder

A plane wave exp(ikx) is incident on the concave side of the sector of a circular cylindrical mirror of radius R. The situation is represented in Figure 5. The center of the mirror has been placed at the origin of a cartesian coordinate system so that the incident wave moves parallel to the x-axis in a positive direction. The angular extent of the mirror is $\alpha + \beta$, where α measures the angle between the x-axis and the end of the sector in the first quadrant, and β measures the angle~~s~~ between the x-axis and the end of the segment in the fourth quadrant.

We shall consider only sectors such that no ray is reflected more than once. It is apparent from Figure 5 that β can be at most $\pi - 3\alpha$.

The phase of the incident wave at a point of reflection on the mirror is $R\cos\theta$, where θ is the angle of incidence; from Figure ~~4~~ 5 it is apparent that the equation of the reflected ray is

$$(37) \qquad (R\sin\theta - y)\,/\,(R\cos\theta - x) \;=\; \tan 2\theta \ ;$$

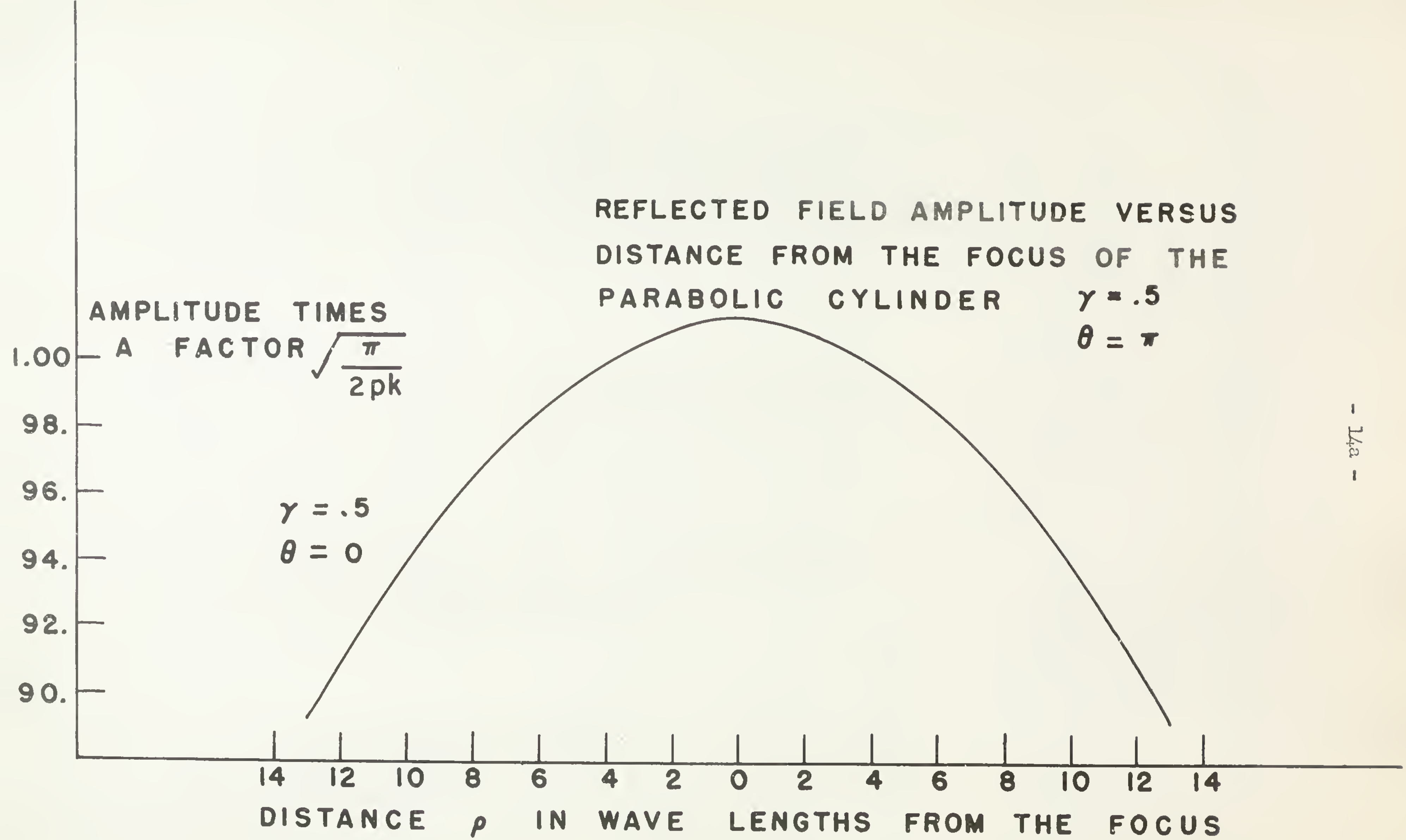
REFLECTED FIELD AMPLITUDE VERSUS
DISTANCE FROM THE FOCUS OF THE
PARABOLIC CYLINDER
γ = .5
θ = π
AMPLITUDE TIMES
A FACTOR √(π/2ρk)
1.00
98.
96.
94.
92.
90.
γ = .5
θ = 0
14 12 10 8 6 4 2 0 2 4 6 8 10 12 14
DISTANCE ρ IN WAVE LENGTHS FROM THE FOCUS

Figure 2

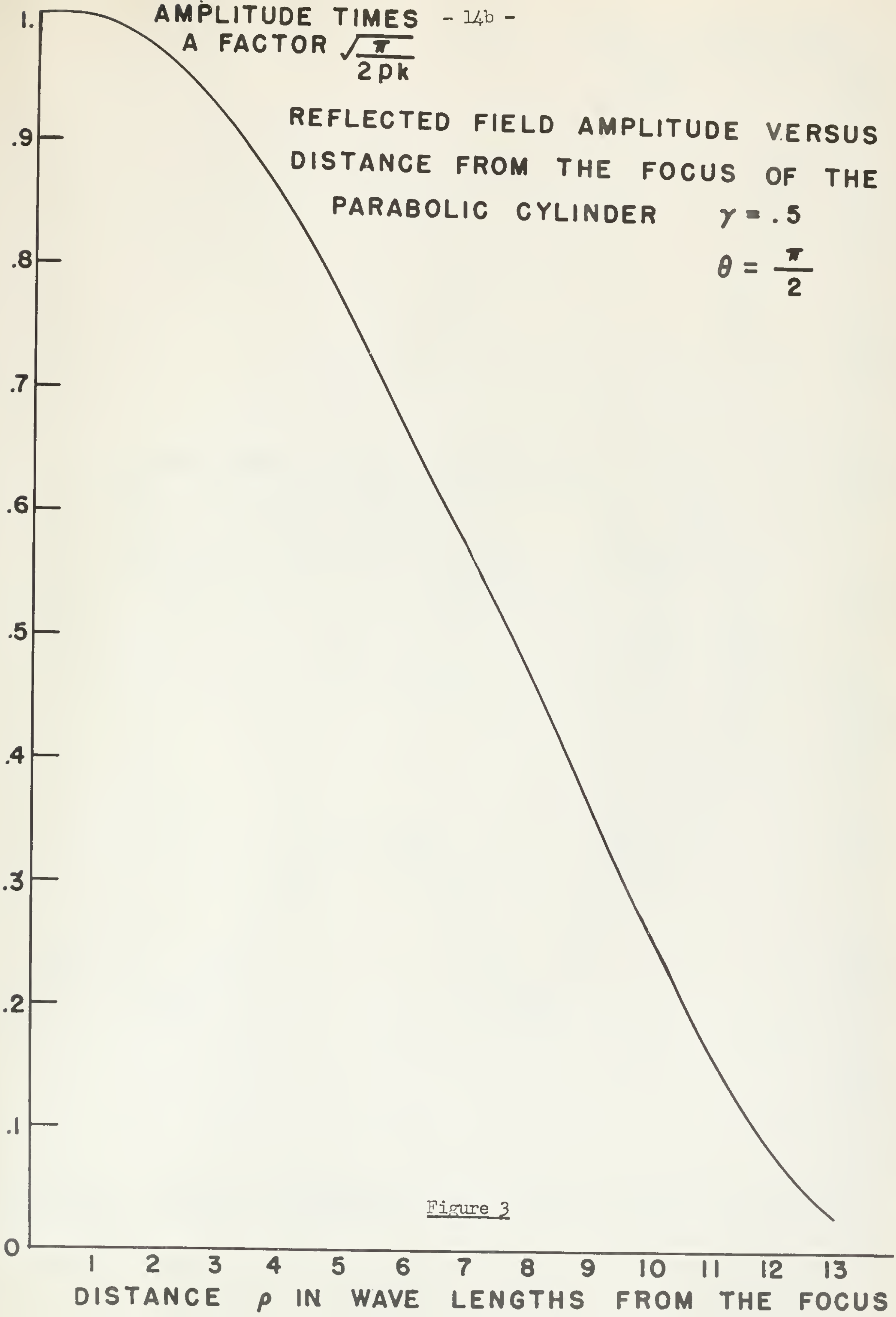
AMPLITUDE TIMES
A FACTOR $\sqrt{\frac{\pi}{2\rho k}}$
REFLECTED FIELD AMPLITUDE VERSUS DISTANCE FROM THE FOCUS OF THE PARABOLIC CYLINDER $\gamma = .5$
$\theta = \frac{\pi}{2}$
1.
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13
DISTANCE ρ IN WAVE LENGTHS FROM THE FOCUS

Figure 3

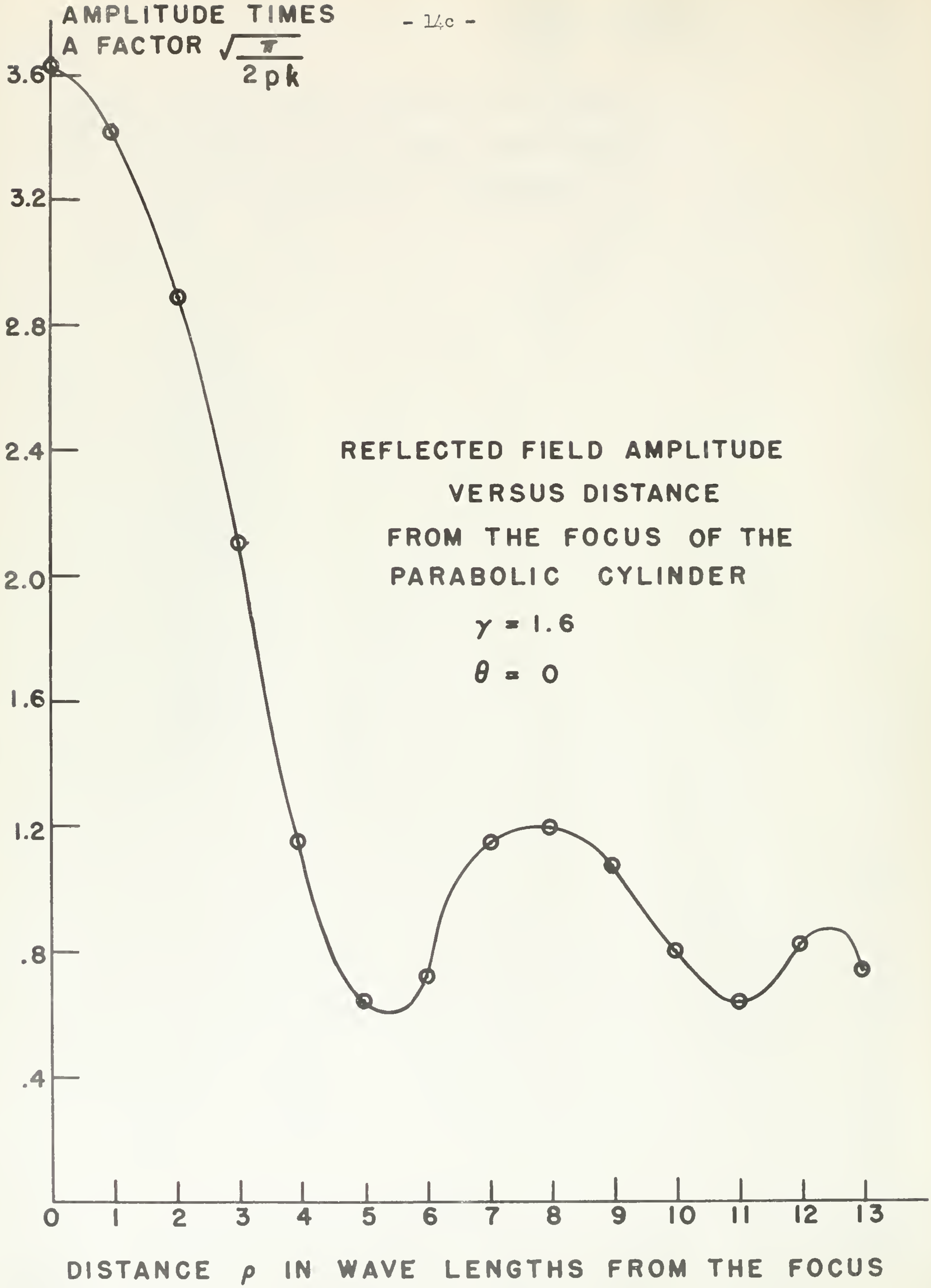
AMPLITUDE TIMES
A FACTOR $\sqrt{\frac{\pi}{2\rho k}}$
3.6
3.2
2.8
2.4
2.0
1.6
1.2
.8
.4
REFLECTED FIELD AMPLITUDE
VERSUS DISTANCE
FROM THE FOCUS OF THE
PARABOLIC CYLINDER
$\gamma = 1.6$
$\theta = 0$
0 1 2 3 4 5 6 7 8 9 10 11 12 13
DISTANCE ρ IN WAVE LENGTHS FROM THE FOCUS

Figure 4

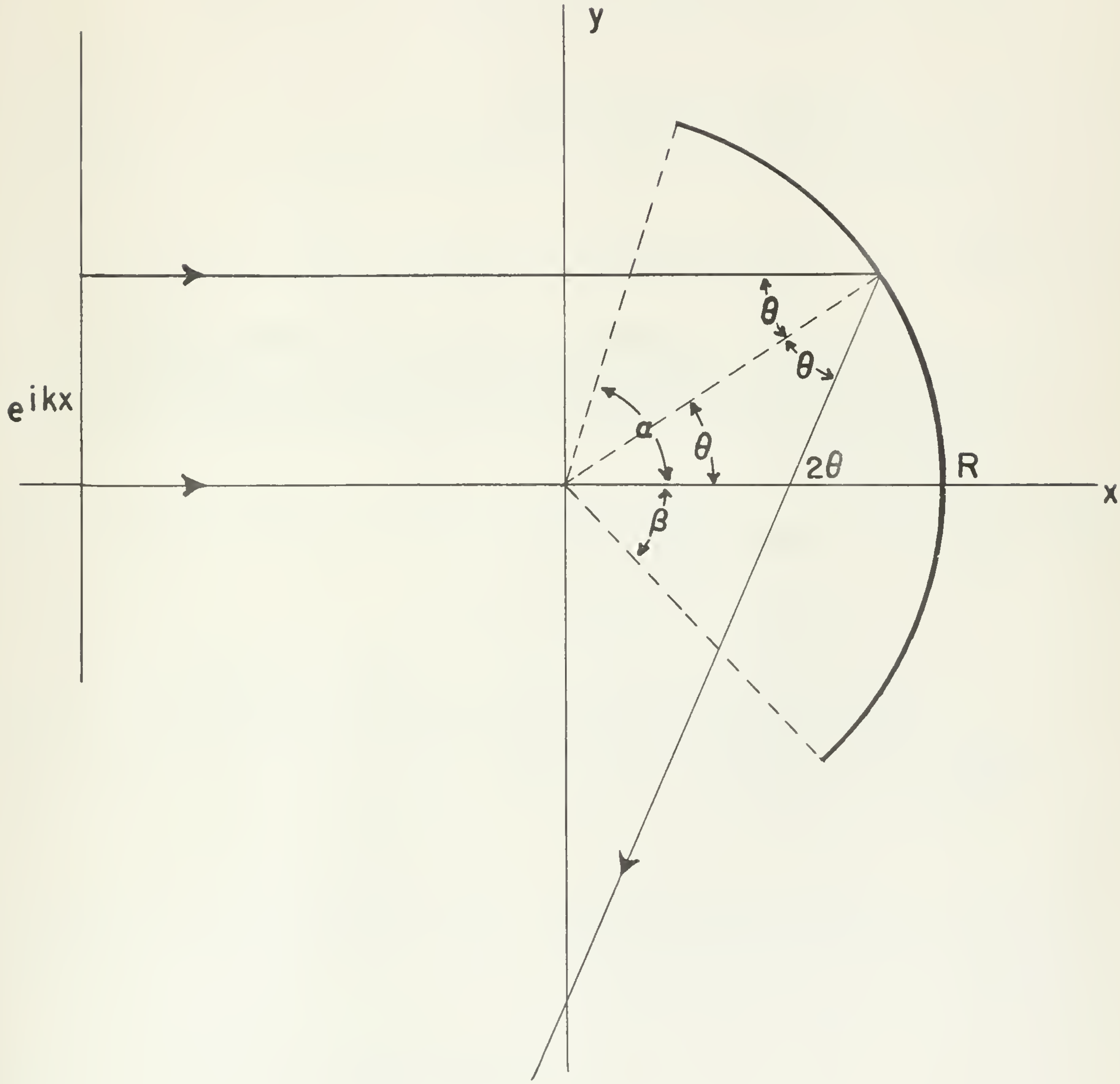

Figure 5

(x,y) is an arbitrary point on the reflected ray.

Let D be the distance the reflected wavefront has traveled from the point of reflection measured along the reflected ray. Then the phase ψ of the reflected wavefront at the point of observation is $D + R\cos\theta$. It is clear from Fig. 5 that

$$(38) \qquad D = (R\cos\theta - x) / \cos 2\theta \,.$$

Thus the total phase at the point of observation is given by

$$(39) \qquad \psi = R\cos\theta + (R\cos\theta - x) / \cos 2\theta \,.$$

By a similar consideration, or from (37), we obtain an expression for ψ involving y. Using this expression and (39) we obtain the following parametric equations for a reflected wavefront i.e., a curve of constant phase ψ :

$$(40) \qquad \begin{aligned} x &= (R\cos\theta - \psi)\cos 2\theta + R\cos\theta \\ y &= (R\cos\theta - \psi)\sin 2\theta + R\sin\theta \,. \end{aligned}$$

We can use (40) to obtain a coordinate system for points (x,y) on a reflected ray in terms of the distance D and the angle of incidence θ by replacing ψ by $R\cos\theta + D$:

$$(41) \qquad \begin{aligned} x &= -D\cos 2\theta + R\cos\theta \\ y &= -D\sin 2\theta + R\sin\theta \,. \end{aligned}$$

Now we observe that the Jacobian of this transformation is $2D - R\cos\theta$, which is zero when $D = \frac{1}{2}R\cos\theta$. This is the distance D^* to the caustic given by our formula (18):

$$(42) \qquad D^* = (2k_c - k_i\cos\theta)^{-1}\cos\theta = (2/R)^{-1}\cos\theta = \frac{1}{2}R\cos\theta \ .$$

We can check this result by calculating the curvature of the reflected wavefront ψ = constant from (40); we obtain the result $K_r = 2(R\cos\theta)^{-1}$ as expected. From our formulas (12) we have for the reflected geometrical optics field the quantity

$$(43) \qquad E_r \sim -\left(1 - \frac{2D}{R\cos\theta}\right)^{-1/2} \exp\left[\,ik(R\cos\theta + D)\,\right]$$

in terms of the coordinate system defined by (41).

At the caustic, where $D = \frac{1}{2}R\cos\theta$, the reflected field takes a different form, which is found from (25):

$$(44) \quad E_r \sim (48)^{\frac{1}{3}}\pi^{-\frac{1}{2}}\,\Gamma\left(\frac{4}{3}\right)\cos\frac{\pi}{6}\cdot k^{\frac{1}{6}}\sqrt{R\cos\theta}\left|\frac{d^3}{d\theta^3}(R\cos\theta + D)\right|^{-\frac{1}{3}}\exp\left[\frac{3}{2}ikR\cos\theta - i\frac{\pi}{4}\right].$$

To calculate the third derivative in this expression we make use of the fact that $\frac{d\psi}{d\theta} = 0$ at the point of reflection and that $D = \frac{1}{2}R\cos\theta$ there. Then

$$(45) \qquad r_o\sin(\theta - \theta_o) = \frac{1}{4}R\sin 2\theta \ .$$

In addition we use the fact that $\frac{d^2\psi}{d\theta^2} = 0$ when the observation point is on the caustic, and obtain

$$(46) \qquad r_o\cos(\theta - \theta_o) = \frac{R}{2}(1 + \sin^2\theta) .$$

Then, using (45) and (46), it is easy to find that when the point of observation is on the caustic,

$$(47) \qquad \frac{d^3\psi}{d\theta^3} = -6R \sin\theta \ .$$

Thus the reflected field at the caustic is

$$(48) \quad E_r \sim 2\pi^{-\frac{1}{2}} \Gamma\left(\tfrac{4}{3}\right) \cos\tfrac{\pi}{6} \cdot (kR)^{\frac{1}{6}} \cos^{\frac{1}{2}}\theta \cdot \sin^{-\frac{1}{3}}\theta \exp\left[\tfrac{3}{2} ikR\cos\theta - i\tfrac{\pi}{4}\right] .$$

This result holds everywhere on the caustic except at $\theta = 0$, where the caustic has a cusp. To obtain the field at this point we need one more derivative of ψ at the cusp; this is given by

$$(49) \qquad \frac{d^4\psi}{d\theta^4} = -6R \ .$$

The reflected field at the cusp is then

$$(50) \quad E_r \sim 2^{\frac{3}{2}} \pi^{-\frac{1}{2}} \Gamma\left(\tfrac{5}{4}\right) \cdot (kR)^{\frac{1}{4}} \exp\left[\tfrac{3}{2} ikR - \tfrac{3}{8} i\pi\right] .$$

In the neighborhood of the caustic curve there is transition between the various forms of the reflected field given by (44), (48) and (50) (49). To obtain this transition form we must return to the integral expression from which the general results were derived. In fact we write for the reflected field

$$(51) \quad E_r \sim (2\pi)^{-\frac{1}{2}} k^{\frac{1}{2}} D^{-\frac{1}{2}} R\cos\theta \exp\left[ik(R\cos\theta + D) - i\tfrac{\pi}{4}\right] \int \exp\Phi \, d\theta \ .$$

where

$$(52)\quad \Phi = \tfrac{1}{2}ik\left\{ (D^{-1} R^2\cos^2\theta - 2R\cos\theta)(\theta'-\theta)^2 + \tfrac{1}{2}D^{-2} R^2\sin 2\theta\,(D - R\cos\theta)(\theta'-\theta)^3 \right\}$$

except in the neighborhood of the cusp. In the neighborhood of the cusp we have

$$\Phi = ik\left\{ \left[\tfrac{1}{2} D^{-1} R^2\cos^2\theta - R\cos\theta\right](\theta'-\theta)^2 + \left[\tfrac{1}{4} D^{-2} R^2\sin 2\theta\,(D-R\cos\theta)\right](\theta'-\theta)^3 \right.$$

$$(53)\qquad + \tfrac{1}{24}\left[R\cos\theta - D^{-1} R\,(R-D\cos\theta) + 4D^{-1} R^2\sin^2\theta - 3D^{-3} R^3 (R-D\cos\theta)^2\right.$$

$$\left.\left. + 18\,D^{-3} R^3\sin^2\theta\,(R-D\cos\theta) - 15D^{-3} R^4\sin^4\theta\right](\theta'-\theta)^4 \right\}.$$

If we use the value of Φ given by (53), the integral in (51) can be evaluated easily in terms of standard functions in the special case where the observation point is the ray reflected through the cusp, the ray being given by $\theta = 0$. On the other hand, if we use the value of Φ given by (52), we can evaluate the integral in (51) with no restriction.

Consider the integral in (51) with Φ given by (52). It has the form

$$(54)\quad I = \int_{-\infty}^{\infty} \exp\left[i\,(\gamma t^3 + \delta t^2)\right] dt = |\gamma|^{-\frac{1}{3}}\int_{-\infty}^{\infty} \exp\left[i\,(s^3 + \delta\gamma^{-\frac{2}{3}} s^2)\right] ds.$$

Let $s = u - \dfrac{\delta\gamma^{-2/3}}{3}$. Then

$$I = 2|\gamma|^{-\frac{1}{3}} \exp\left[\frac{2i\delta^3}{27\gamma^2}\right]\int_0^\infty \cos\left(\frac{u^3 - \delta^2 u}{3\gamma^{4/3}}\right) du,$$

(55)

$$= 2|\gamma|^{-\frac{1}{3}} \exp\left[\frac{2i\delta^3}{27\gamma^2}\right] Ci_3\left(\frac{-\delta^2}{9\gamma^{4/3}}\right) ,$$

where $Ci_n(a)$ is Hardy's generalized Airy integral given by Watson [6] :

(56) $$Ci_3(a) = \left(\frac{\pi a^{1/2}}{6\sin(\pi/6)}\right)\left[J_{-\frac{1}{3}}(2a^{3/2}) - J_{\frac{1}{3}}(2a^{3/2})\right] .$$

According to (52) we have

$$\gamma = (kR^2 \sin 2\theta / 4D^2)(D - R\cos\theta)$$

(57)

$$\delta = \frac{1}{2}k\left[R^2\cos^2\theta / D) - 2R\cos\theta\right] ,$$

and for the reflected wave field in the neighborhood of the caustic

(58) $$E_r \sim (2k/\pi D)^{\frac{1}{2}}|\gamma|^{-\frac{1}{3}} R\cos\theta \exp\left[ik(R\cos\theta + D) + i\left(\frac{2\delta^3}{27\gamma^2} - \frac{\pi}{4}\right)\right] Ci_3\left(\frac{-\delta^2}{3\gamma^{4/3}}\right) .$$

It is understood of course that (58) holds only when the point of observation is away from the cusp.

If the point of observation is in the neighborhood of the cusp and on the line $\theta = 0$, we can use (51) and (53) to obtain the reflected wave field in a form similar to (58). Here the integral on the right of (51) has the form

$$(59)\quad I = \int_{-\infty}^{\infty} \exp\left[-i(\alpha\theta'^4 + \beta\,\theta'^2) \right] d\theta' = \alpha^{-\frac{1}{4}} \int_{-\infty}^{\infty} \exp\left[-i(t^4 + \beta\alpha^{-\frac{1}{2}} t^2)\right] dt.$$

We assume $\alpha > 0$ in the region considered, since $D \sim \frac{1}{2}R$ there. Then

$$(60)\quad I = a\alpha^{-\frac{1}{4}} \exp(i\beta^2/8\alpha) \left\{ C_{i4}\left(\tfrac{1}{4}\beta\alpha^{-\frac{1}{2}}\right) - i\, S_{i4}\left(\tfrac{1}{4}\beta\alpha^{-\frac{1}{2}}\right) \right\}.$$

Thus

$$(61)\quad E_r \sim (2k/\pi D)^{\frac{1}{2}} R\alpha^{-\frac{1}{4}} \exp\left[ik(R + D) + i(\beta^2/8\alpha - \pi/4)\right] \left\{ C_{i4}\left(\tfrac{1}{4}\beta\alpha^{-\frac{1}{2}}\right) - iS_{i4}\left(\tfrac{1}{4}\beta\alpha^{-\frac{1}{2}}\right) \right\}$$

with

$$(62)\quad \alpha = \frac{1}{24}\left[RD^{-1}(R - D) - R + 3R^3 D^{-3} (R - D)^2 \right]$$

$$\beta = R - \frac{1}{2} R^2 D^{-1}.$$

Again the functions C_{i4} and S_{i4} are Hardy's generalized Airy integrals[6] :

$$C_{in}(\alpha) = \pi\alpha^{\frac{1}{2}} \left[2n \sin (\pi/2n)\right]^{-1} \left\{ J_{-1/n}(2\alpha^{n/2}) - J_{1/n}(2\alpha^{n/2}) \right\}$$

(63)

$$S_{in}(\alpha) = \pi\alpha^{\frac{1}{2}} \left[2n \cos (\pi/2n)\right]^{-1} \left\{ J_{-1/n}(2\alpha^{n/2}) + J_{1/n}(2\alpha^{n/2}) \right\} .$$

Equation (61) gives the reflected field in the neighborhood of the cusp provided that the observation point is on the axis of the circular mirror.

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